

De Rham cohomology of CP^1 model with Hopf term

Soon-Tae Hong*

*Research Institute for Basic Sciences and Department of Science Education,
Ewha Womans University, Seoul 120-750, Korea*

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We investigate the CP^1 model possessing the Hopf term which respects the second class constraints and admits the well defined BRST operator Q . Using the operator Q , we explicitly construct its de Rham cohomology group by deriving the ensuing quotient group via both the collections of all Q -closed and Q -exact ghost number p -forms. Moreover, we study the CP^1 model without the Hopf term to evaluate the ensuing effect of the Hopf term on the cohomology group. We find that the Hopf term effects on the de Rham cohomology originate from the non-compact Hilbert space modified by this Hopf term.

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The CP^1 model is a complex extension of the Heisenberg $O(3)$ model. The $O(3)$ model emerges in various physical applications from high energy physics [1] to condensed matter physics [2]. Recently, there have been some progresses in knotted solitons [3]. The action of the CP^1 model with the Hopf term possesses a desirable manifest locality since the Hopf term has a local integral representation in terms of the physical fields of the CP^1 model [4]. Our aim is to investigate the Hopf term effects on de Rham cohomology [5–9] involved in the CP^1 model possessing the this term. To do this at first we will recapitulate shortly the first class formalism of the CP^1 model without the Hopf term [10]. The CP^1 model without the Hopf term is defined by the Lagrangian,

$$L_0 = \int d^2x [(\partial_\mu \xi_\alpha^*)(\partial^\mu \xi_\alpha) - (\xi_\alpha^* \partial_\mu \xi_\alpha)(\xi_\beta \partial^\mu \xi_\beta^*)]. \quad (1)$$

Here $\xi_\alpha = (\xi_1, \xi_2)$ is a multiplet of complex scalar fields satisfying a constraint

$$\omega_1 = |\xi_\alpha|^2 - 1 \approx 0, \quad (2)$$

and thus the CP^1 model becomes a second class constrained Hamiltonian system. Exploiting the Lagrangian (1), we find that the canonical Hamiltonian is given by

$$H_0 = \int d^2x [|\pi_\alpha|^2 + |\partial_i \xi_\alpha|^2 - (\xi_\alpha^* \partial_i \xi_\alpha)(\xi_\beta \partial_i \xi_\beta^*)], \quad (3)$$

where $(\pi_\alpha, \pi_\alpha^*)$ are the canonical momenta conjugate to the complex scalar fields $(\xi_\alpha, \xi_\alpha^*)$. By implementing the Dirac algorithm [11] we find that, together with Eq. (2), our Hamiltonian system is subject to another second class constraint

$$\omega_2 = \xi_\alpha^* \pi_\alpha^* + \xi_\alpha \pi_\alpha \approx 0. \quad (4)$$

Introducing two canonically conjugate Stückelberg fields (θ, π_θ) we obtain the first class Hamiltonian

$$\tilde{H}_0 = \int d^2x \left[|\pi_\alpha - \frac{1}{2} \xi_\alpha^* \pi_\theta|^2 R + |\partial_i \xi_\alpha|^2 \frac{1}{R} - (\xi_\alpha^* \partial_i \xi_\alpha)(\xi_\beta \partial_i \xi_\beta^*) \frac{1}{R^2} \right], \quad (5)$$

where $R = |\xi_\alpha|^2 / (|\xi_\alpha|^2 + 2\theta)$.

Up to now, we briefly summarize the first class formalism of the CP^1 model appeared in Ref. [10]. Now we are ready to investigate the cohomology group of the CP^1 model which is our main theme. Introducing two canonical sets of ghost and anti-ghost fields together with auxiliary fields (c^i, \bar{p}_i) , (p^i, \bar{c}_i) , (n^i, b_i) ($i = 1, 2$), we define the BRST operator for our constraint algebra

$$Q = \int d^2x (c^i \tilde{\omega}_i + p^i b_i), \quad (6)$$

*Electronic address: soonhong@ewha.ac.kr

from which we find

$$\delta_Q \tilde{H}_0 = 0. \quad (7)$$

Here one notes that $\tilde{\omega}_i$ ($i = 1, 2$) are the first class constraints of the previous second class ones ω_i in Eqs. (2) and (4). Moreover, one can readily shows that the BRST charge Q is nilpotent: $Q^2 = 0$.

We choose the unitary gauge with $(\chi^1, \chi^2) = (\omega_1, \omega_2)$ by selecting the gauge fixing functional

$$\psi = \int d^2x (\bar{c}_i \chi^i + \bar{p}_i n^i). \quad (8)$$

We then have

$$\delta_Q \delta_Q \psi = 0, \quad (9)$$

which follows from the nilpotency of the charge Q . Now the gauge fixed BRST invariant Hamiltonian is given by

$$\begin{aligned} H_{eff} &= \tilde{H} - \delta_Q \psi, \\ \tilde{H} &= \tilde{H}_0 + \int d^2x \left(\frac{1}{2} \pi_\theta \tilde{\omega}_2 - c^1 \bar{p}_2 \right). \end{aligned} \quad (10)$$

Here one notes that in order to guarantee the BRST invariance of H_{eff} we have included in H_{eff} the Q -exact term, and in \tilde{H} the term possessing π_θ and the Faddeev-Popov ghost term [12]. Moreover, the term $\delta_Q \psi$ fixes the particular unitary gauge corresponding to the fixed point ($\theta = 0, \pi_\theta = 0$) in the gauge degrees of freedom associated with two-dimensional manifold described by the internal phase space coordinates (θ, π_θ) , which are two canonically conjugate Stückelberg fields.

Next, we introduce the BRST operator via

$$Q : \alpha_p \rightarrow \alpha_{p+1}, \quad (11)$$

where α_p is a ghost number p -form. We then can construct the p -th de Rham cohomology group $C^p(M, R)$ of the manifold M and the field of real number R with the following quotient group [5–9]

$$C^p(M, R) = \frac{Z^p(M, R)}{B^p(M, R)}. \quad (12)$$

Here $Z^p(M, R)$ are the collection of all Q -closed ghost number p -forms α_p for which $Q\alpha_p = 0$, and $B^p(M, R)$ are the collection of all Q -exact ghost number p -forms α_p for which $\alpha_p = Q\alpha_{p-1}$.

The Hamiltonians \tilde{H} and H_{eff} are readily shown to be the ghost number 0-forms, and they are Q -closed as in the case of $\delta_Q \psi$. These Hamiltonians thus can be used to define the $Z^0(M, R)$. Here M is the non-compact Hilbert space of the CP^1 model and R is the real number field. Next, we notice that ψ is the ghost number (-1) -form and $Q\psi$ is Q -exact ghost number 0-form, so that we can define the $B^0(M, R)$. With these $Z^0(M, R)$ and $B^0(M, R)$, we construct the 0-th de Rham cohomology group $C^0(M, R)$:

$$C^0(M, R) = \frac{Z^0(M, R)}{B^0(M, R)}, \quad \text{for } CP^1 \text{ model.} \quad (13)$$

It is interesting to note that the ghost number 0-form H_{eff} is deformed into the other ghost number 0-form \tilde{H} . Namely, H_{eff} is homologous to \tilde{H} under the BRST transformation Q , or $H_{eff} \sim \tilde{H}$, since $Q\psi = \tilde{H} - H_{eff}$. These features also appear [8, 9] even in the 't Hooft-Polyakov monopole [13, 14], which is physics-wise different from the CP^1 model.

Now we generalize the above results to the CP^1 model with Hopf term. The new Lagrangian is given by

$$L_0^H = L_0 - \int d^2x \frac{\Theta}{8\pi^2} \epsilon^{\mu\nu\rho} (\xi_\alpha^* \partial_\mu \xi_\alpha - \partial_\mu \xi_\alpha^* \xi_\alpha) \partial_\nu \xi_\beta^* \partial_\rho \xi_\beta. \quad (14)$$

Here one notes that our physical system with the Hopf term also respects the constraints ω_i ($i = 1, 2$) in Eqs. (2) and (4). Following the same algorithm discussed above, we arrive at the first class Hamiltonian with Hopf term [15]

$$\tilde{H}_0^H = \int d^2x \left[|\pi_\alpha^H|^2 - \frac{1}{2} \xi_\alpha^* \pi_\theta + \frac{\Theta}{8\pi^2 R^2} \pi_\alpha^\Theta |\pi_\alpha^H|^2 R + |\partial_i \xi_\alpha|^2 \frac{1}{R} - (\xi_\alpha^* \partial_i \xi_\alpha) (\xi_\beta \partial_i \xi_\beta^*) \frac{1}{R^2} \right]. \quad (15)$$

Here π_α^H originates from the canonical momenta conjugate to the complex scalar fields ξ_α associated with the CP^1 Lagrangian with the Hopf term (14), while π_α^Θ is related with the difference between π_α and π_α^H . Introducing the ghost and anti-ghost fields together with their auxiliary fields as above, we obtain the identities

$$\begin{aligned} H_{eff}^H &= \tilde{H}^H - \delta_Q \psi, \\ \tilde{H}^H &= \tilde{H}_0^H + \int d^2x \left(\frac{1}{2} \pi_\theta \tilde{\omega}_2 - c^1 \bar{p}_2 \right). \end{aligned} \quad (16)$$

Resorting to the routine procedure above, we proceed to formulate the cohomology group as follows

$$C^0(M^H, R) = \frac{Z^0(M^H, R)}{B^0(M^H, R)}, \quad \text{for } CP^1 \text{ model with Hopf term.} \quad (17)$$

Here one notes that the non-compact Hilbert space M is modified into the non-compact M^H due to the Hopf term in the CP^1 model.

In conclusion, we have studied the CP^1 model possessing the Hopf term which respects the second class constraints and is related with the BRST operator Q . Exploiting Q , we have explicitly constructed its de Rham cohomology group by deriving the ensuing quotient group via both the collections of all Q -closed and Q -exact ghost number p -forms. Moreover, we have introduced the CP^1 model without the Hopf term to investigate the ensuing effect of the Hopf term on the cohomology group. We have concluded that the Hopf term effects on the de Rham cohomology originate from the non-compact Hilbert space modified by this Hopf term. This is one of main results of the algebraic topology aspects of the CP^1 model with the Hopf term.

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